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Saint-Venant end effects for plane deformations of linear piezoelectric solids

Alessandra Borrelli [a], Cornelius O. Horgan [b,*], M. Cristina Patria [c]

[a] *Dipartimento di Matematica, Università di Ferrara, I-44100 Ferrara, Italy*
[b] *Structural and Solid Mechanics Program, Department of Civil Engineering, University of Virginia, Charlottesville, VA 22904, USA*
[c] *Dipartimento di Ingegneria Civile, Università di Udine, I-33100 Udine, Italy*

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Dedicated to J.G. Simmonds on the occasion of his 70th birthday

Abstract

The recent developments in smart structures technology have stimulated renewed interest in the fundamental theory and applications of linear piezoelectricity. In this paper, we investigate the decay of Saint-Venant end effects for plane deformations of a piezoelectric semi-infinite strip. First of all, we develop the theory of plane deformations for a general anisotropic linear piezoelectric solid. Just as in the mechanical case, not all linear homogeneous anisotropic piezoelectric cylindrical solids will sustain a non-trivial state of plane deformation. The governing system of *four* second-order partial differential equations for the two in-plane displacements and electric potential are overdetermined in general. Sufficient conditions on the elastic and piezoelectric constants are established that do allow for a state of plane deformation. The resulting traction boundary-value problem with prescribed surface charge is an oblique derivative boundary-value problem for a coupled elliptic system of *three* second-order partial differential equations. The special case of a piezoelectric material transversely isotropic about the poling axis is then considered. Thus the results are valid for the hexagonal crystal class 6*mm*. The geometry is then specialized to be a two-dimensional semi-infinite strip and the poling axis is the axis transverse to the longitudinal direction. We consider such a strip with sides traction-free, subject to zero surface charge and self-equilibrated conditions at the end and with tractions and surface charge assumed to decay to zero as the axial variable tends to infinity. A formulation of the problem in terms of an Airy-type stress function and an induction function is adopted. The governing partial differential equations are a coupled system of a *fourth* and *third-order* equation for these two functions. On seeking solutions that exponentially decay in the axial direction one obtains an *eigenvalue problem* for a coupled system of *fourth* and *second-order* ordinary differential equations. This problem is the piezoelectric analog of the well-known eigenvalue problem arising in the case of an anisotropic elastic strip. It is shown that the problem can be uncoupled to an eigenvalue problem for a *single sixth-order* ordinary differential equation with complex eigenvalues characterized as roots of transcendental equations governing symmetric and

* Corresponding author. Tel.: +1 434 924 7230; fax: +1 434 982 2951.
E-mail address: coh8p@virginia.edu (C.O. Horgan).

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anti-symmetric deformations and electric fields. Assuming completeness of the eigenfunctions, the rate of decay of end effects is then given by the real part of the eigenvalue with smallest positive real part. Numerical results are given for PZT-5H, PZT-5, PZT-4 and Ceramic-B. It is shown that *end effects for plane deformations of these piezoceramics penetrate further into the strip than their counterparts for purely elastic isotropic materials*.

Keywords: Saint-Venant end effects; Plane deformations; Linear piezoelectric materials

1. Introduction

The recent developments in smart structures technology have stimulated renewed interest in the fundamental theory and applications of linear piezoelectricity (see, e.g., Eringen and Maugin, 1990; Ikeda, 1996 for a comprehensive treatment of the basic theory). In this paper, we are concerned with a specific issue for linear homogeneous piezoelectric solids, namely investigation of the decay of Saint-Venant end effects for plane deformations of a semi-infinite strip.

Saint-Venant's principle and related issues for elasticity theory have been extensively studied (for reviews, see, e.g., Horgan and Knowles, 1983; Horgan, 1989, 1996a,b; Flavin and Rionero, 1996; Horgan and Carlsson, 2000) but it is only relatively recently that analogous questions for piezoelectricity have been investigated. The three-dimensional piezoelectric cylinder problem has been examined by Batra and Yang (1995), Batra and Zhong (1996), Bisegna (1998) and Borrelli and Patria (1999) while anti-plane shear problems have been studied by the present authors in Borrelli et al. (2001, 2002, 2003, 2004). The plane problem has been investigated by Fan (1995), by Ruan et al. (2000) and by Tarn and Huang (2002), who also consider multilayered piezoelectric laminates.

In the next section, we develop the theory of plane deformations for a general anisotropic linear piezoelectric solid. Just as in the mechanical case (see, e.g., Horgan and Miller, 1994; Horgan, 1995; Ting, 1996), not all linear homogeneous anisotropic piezoelectric cylindrical solids will sustain a non-trivial state of plane deformation. The governing system of *four* second-order partial differential equations for the two in-plane displacements and electric potential are overdetermined in general. Sufficient conditions on the elastic and piezoelectric constants are established that do allow for a state of plane deformation. These are generalizations of the results of Horgan and Miller (1994) and Ting (1996) for the purely mechanical case. The resulting traction boundary-value problem with prescribed surface charge is an oblique derivative problem (i.e., the boundary conditions are not simply for the outward normal derivative) for a coupled system of *three* second-order partial differential equations. Sufficient conditions for ellipticity of the system are given. In Section 3, the special case of a piezoelectric material transversely isotropic about the poling axis is considered. Thus the results subsequently obtained in this paper are valid for the hexagonal crystal class 6*mm* and numerical results are given later on for PZT-5H, PZT-5, PZT-4 and Ceramic-B. In Section 4, the geometry is specialized to be a two-dimensional semi-infinite strip and the poling axis is the axis transverse to the longitudinal direction. To examine the extent of Saint-Venant end effects, it is sufficient to consider such a strip with sides traction-free and subject to zero surface charge and "self-equilibrated" conditions at the end. The tractions and surface charge are assumed to decay to zero as the axial variable tends to infinity. This problem is investigated in detail in Sections 3 and 4 for the 6*mm* strip. A formulation in terms of an Airy-type stress function and an induction function is adopted. The governing partial differential equations (see (25)) are a coupled system of a *fourth* and *third-order* equation for these two functions. In Section 4, solutions are sought in the form of exponential functions of the axial variable multiplied by unknown functions of the transverse variable. This, together with the boundary conditions, leads to an *eigenvalue problem* for a coupled system of *fourth* and *second-order* ordinary differential equations. This problem is the piezoelectric analog of the well-known eigenvalue problem arising in the case of an anisotropic elastic strip (see, e.g., Choi and Horgan, 1977; Crafter et al., 1993; Ting, 1996), which in turn is a generalization of the celebrated Fadle-Papkovich problem for the isotropic solid. It turns out to be convenient to uncouple the problem to an eigenvalue problem for a *single sixth-order* ordinary differential equation (see (33)). In Section 5, it is shown how the complex eigenvalues are characterized as roots of transcendental equations governing symmetric and anti-symmetric deformations and electric fields. Assuming completeness of the eigenfunctions, the rate of decay of end effects is then given by the real part of the eigenvalue with smallest positive real part. In Sections 6 and 7, numerical results are given for PZT-5H, PZT-5, PZT-4 and Ceramic-B. It is shown that *end effects for plane deformations of these piezoceramics penetrate further into the strip than their counterparts for purely isotropic elastic materials,* confirming results of Borrelli et al. (2001, 2002, 2003, 2004) for anti-plane shear and consistent with results of Ruan et al. (2000) and of Tarn and Huang (2002) for the plane problem.

2. Plane deformations for linear anisotropic piezoelectric materials

Let Ω be an infinite cylinder $\Omega = \{\mathbf{x} \in \mathbf{R}^3 : (x_1, x_3) \in \Sigma\}$ where Σ denotes the plane simply-connected cross-section with boundary $\partial\Sigma$ assumed sufficiently smooth. Let Ω be occupied by a homogeneous anisotropic linearly piezoelectric material in equilibrium and suppose that the material is polarized along the x_3 axis. On using the usual Cartesian tensor notation, the governing equations (in the absence of body forces and free electric volume charge) are (Eringen and Maugin, 1990; Ikeda, 1996)

$$\begin{aligned} &T_{ij,j} = 0, \quad D_{i,i} = 0, \\ &T_{ij} = C_{ijkl}u_{k,l} + e_{kij}\varphi_{,k}, \quad D_i = e_{ijk}u_{j,k} - \varepsilon_{ij}\varphi_{,j}, \end{aligned} \tag{1}$$

where T_{ij}, D_i denote, respectively, the components of the stress tensor and of the electric displacement vector, u_i denote the components of the mechanical displacement vector field and φ is the electric potential from which the electric field $\mathbf{E}$ is given by $\mathbf{E} = -\nabla\varphi$. The elastic, piezoelectric and electric permittivity constants have the following symmetry properties:

$$\begin{aligned} &C_{ijk\ell} = C_{jik\ell} = C_{k\ell ij}, \\ &e_{ijk} = e_{ikj}, \quad \varepsilon_{ij} = \varepsilon_{ji}. \end{aligned} \tag{2}$$

We now suppose that Ω is subjected to prescribed surface tractions $t_i^* = T_{ij}n_j$ and surface charge $D^* = D_in_i$ on its lateral surface $\partial\Omega$ ($\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$) of the form

$$t_2^* = 0, \quad t_1^* = t_1^*(x_1, x_3), \quad t_3^* = t_3^*(x_1, x_3), \quad D^* = D^*(x_1, x_3). \tag{3}$$

The special form of the boundary conditions (3) would be expected to give rise to a deformation $\mathbf{u}$ and an electric potential φ of the form

$$u_2 = 0, \quad u_1 = u_1(x_1, x_3), \quad u_3 = u_3(x_1, x_3), \quad \varphi = \varphi(x_1, x_3), \quad \forall (x_1, x_3) \in \Sigma. \tag{4}$$

A mechanical deformation $\mathbf{u}$ of the above form is called a *plane deformation* (see, e.g., Ting, 1996) and (u_1, u_3, φ) is a plane state for the piezoelectric body. Of course, other combinations of boundary conditions could also be considered but we shall not pursue this here.

More generally, if one assumes from the outset a *generalized plane deformation*, i.e.,

$$u_i = u_i(x_1, x_3), \quad i = 1, 2, 3, \quad \varphi = \varphi(x_1, x_3), \tag{5}$$

then one can prove that the boundary conditions (3) give rise to a *plane deformation* under suitable hypotheses on the material constants C_{ijkl}, e_{kij}. In fact, under the assumptions (5), the governing equations become

$$
\begin{aligned}
&C_{2\alpha\beta\gamma}u_{\beta,\gamma\alpha} + C_{2\alpha2\beta}u_{2,\beta\alpha} + e_{\beta2\alpha}\varphi_{,\beta\alpha} = 0,\\
&C_{\delta\alpha\beta\gamma}u_{\beta,\gamma\alpha} + C_{\delta\alpha2\beta}u_{2,\beta\alpha} + e_{\beta\delta\alpha}\varphi_{,\beta\alpha} = 0,\\
&e_{\alpha\beta\gamma}u_{\beta,\gamma\alpha} + e_{\alpha2\beta}u_{2,\beta\alpha} - \varepsilon_{\beta\alpha}\varphi_{,\beta\alpha} = 0 \quad \text{on } \Sigma,
\end{aligned}
\tag{6}
$$

with δ, α, β, $\gamma = 1{,}3$. *In the sequel Greek indices will always have the values 1, 3.*

If we suppose that

$$
C_{2\alpha\beta\gamma} = 0, \quad e_{\alpha2\beta} = 0,
\tag{7}
$$

then (6) reduce to

$$
\begin{aligned}
&C_{2\alpha2\beta}u_{2,\beta\alpha} = 0,\\
&C_{\delta\alpha\beta\gamma}u_{\beta,\gamma\alpha} + e_{\beta\delta\alpha}\varphi_{,\beta\alpha} = 0,\\
&e_{\alpha\beta\gamma}u_{\beta,\gamma\alpha} - \varepsilon_{\alpha\beta}\varphi_{,\beta\alpha} = 0 \quad \text{on } \Sigma.
\end{aligned}
\tag{8}
$$

With Eqs. (8) we associate the boundary conditions (3), i.e.

$$
\begin{aligned}
&C_{2\alpha2\beta}u_{2,\beta}n_\alpha = t_2^* = 0, \quad (C_{\delta\alpha\beta\gamma}u_{\beta,\gamma} + e_{\beta\delta\alpha}\varphi_{,\beta})n_\alpha = t_\delta^*,\\
&(e_{\alpha\beta\gamma}u_{\beta,\gamma} - \varepsilon_{\alpha\beta}\varphi_{,\beta})n_\alpha = D^* \quad \text{on } \partial\Sigma.
\end{aligned}
\tag{9}
$$

Eq. $(8)_1$ together with the boundary condition $(9)_1$ gives

$$
\int_\Sigma C_{2\alpha2\beta}u_{2,\alpha}u_{2,\beta}\,\mathrm{d}\Sigma = 0.
$$

If we assume that the elasticity tensor $\mathbf{C}$ is positive definite, we deduce that u_2 is constant and so, without loss of generality, we can assume that this constant is zero. Thus, the *out-of-plane displacement has been shown to be zero* if the conditions (7) hold. Consequently, under these conditions, which are *stronger* than those assumed by Tarn and Huang (2002), a *pure plane state of deformation can occur*. Thus, the conditions (7) are *sufficient* in order for a plane state (4) to exist and the governing equations (8) reduce to

$$
\begin{aligned}
&C_{\delta\alpha\beta\gamma}u_{\beta,\gamma\alpha} + e_{\beta\delta\alpha}\varphi_{,\beta\alpha} = 0,\\
&e_{\alpha\beta\gamma}u_{\beta,\gamma\alpha} - \varepsilon_{\alpha\beta}\varphi_{,\beta\alpha} = 0 \quad \text{on } \Sigma,
\end{aligned}
\tag{10}
$$

subject to the boundary conditions

$$
\begin{aligned}
&(C_{\delta\alpha\beta\gamma}u_{\beta,\gamma} + e_{\beta\delta\alpha}\varphi_{,\beta})n_\alpha = t_\delta^*,\\
&(e_{\alpha\beta\gamma}u_{\beta,\gamma} - \varepsilon_{\alpha\beta}\varphi_{,\beta})n_\alpha = D^* \quad \text{on } \partial\Sigma.
\end{aligned}
\tag{11}
$$

Moreover, from (10), (11) and use of the divergence theorem, it follows that t_α^* and D^* must satisfy the compatibility conditions

$$
\int_{\partial\Sigma} t_\delta^*\,\mathrm{d}s = 0, \quad \int_{\partial\Sigma} D^*\,\mathrm{d}s = 0
\tag{12}
$$

in order for a solution to (10), (11) to exist. An analog of the foregoing result for the piezoelectric anti-plane shear problem was obtained by Borrelli et al. (2002).

The problem (10), (11) is an oblique derivative boundary-value problem for a coupled system of three *second-order* partial differential equations for the three unknowns u_1, u_3 and φ. As we are concerned with equilibrium problems only, we shall assume that (10) is an *elliptic* system. It can be shown that this is the case if $\mathbf{C}$ and $\boldsymbol{\varepsilon}$ satisfy the following conditions

$$\begin{aligned} &C_{1111} > 0, \quad C_{1111}C_{3333} - C_{1133}^2 > 0, \\ &C_{1313}(C_{1111}C_{3333} - C_{1133}^2) + 2C_{1133}C_{3313}C_{1131} - C_{1131}^2C_{3333} - C_{3313}^2C_{1111} > 0, \\ &\varepsilon_{11} > 0, \quad \varepsilon_{11}\varepsilon_{33} - \varepsilon_{13}^2 > 0, \end{aligned} \tag{13}$$

which we shall assume to hold henceforth. The conditions (13) are guaranteed to hold if we make the usual positive-definiteness assumptions on $\mathbf{C}$ and $\boldsymbol{\varepsilon}$. Thus all known results on existence, uniqueness and regularity of solutions to elliptic systems are applicable to the present problem.

On using (7) in $(1)_3$, $(1)_4$, we find that the stresses and electric displacement vector are given by

$$\begin{aligned} &T_{2\alpha} = T_{\alpha 2} = 0, \\ &T_{\alpha\beta} = C_{\alpha\beta\gamma\delta}u_{\gamma,\delta} + e_{\gamma\alpha\beta}\varphi_{,\gamma}, \quad T_{22} = C_{22\gamma\delta}u_{\gamma,\delta} + e_{\alpha 22}\varphi_{,\alpha}, \end{aligned} \tag{14}$$

and

$$D_\alpha = e_{\alpha\beta\gamma}u_{\beta,\gamma} - \varepsilon_{\alpha\beta}\varphi_{,\beta}, \quad D_2 = e_{2\alpha\beta}u_{\alpha,\beta} - \varepsilon_{2\alpha}\varphi_{,\alpha} \tag{15}$$

respectively. We observe from (14) and (15) that a non-zero axial stress T_{22} and axial electric displacement D_2 can occur, depending on the mechanical anisotropy and piezoelectric and electric permittivity constants.

As observed by Horgan and Miller (1994) for homogeneous *elastic* solids, *anti-plane* shear deformations $u_1 = u_3 = 0$, $u_2 = u_2(x_1, x_3)$ and the corresponding *plane strain deformations* $u_1 = u_1(x_1, x_3)$, $u_3 = u_3(x_1, x_3)$, $u_2 = 0$ can occur if $C_{2\alpha\beta\gamma} = 0$. Therefore, in elasticity, these conditions assure that *both* deformations can occur and that these deformations are *uncoupled*. In piezoelectric solids (see, e.g., Borrelli et al., 2002) the conditions $C_{2\alpha\beta\gamma} = 0$, $e_{\alpha\beta\gamma} = 0$ ensure the existence of the anti-plane state $u_1 = u_3 = 0$, $u_2 = u_2(x_1, x_3)$, $\varphi = \varphi(x_1, x_3)$. We saw above that sufficient conditions for existence of the plane deformation $u_1 = u_1(x_1, x_3)$, $u_3 = u_3(x_1, x_3)$, $u_2 = 0$, $\varphi = \varphi(x_1, x_3)$ are the conditions (7). Thus, *in contrast with elasticity*, the conditions on the piezoelectric constants to ensure the existence of both states are *different* from each other.

We note that plane deformations of the form $u_1 = u_1(x_1, x_2)$, $u_2 = u_2(x_1, x_2)$, $u_3 = 0$, $\varphi = \varphi(x_1, x_2)$ can occur only for very few piezoelectric crystal classes (actually $\bar{6}, \bar{6}m2$), while the plane deformations considered here, i.e., deformations with the x_3-axis as polarization direction, can occur for many crystal classes (6*mm*, 4*mm*, 4, 6 and so on).

3. Plane deformations for the hexagonal crystal class 6*mm* (transversely isotropic)

Consider a piezoelectric material transversely isotropic about the poling axis, which we take as the x_3-axis. An example of such a material is lead–zirconate–titanate (e.g., PZT-5H). In our analysis it is convenient to rewrite (1_3), (1_4) in the form (cf., Eringen and Maugin, 1990)

$$\begin{aligned} &e_{ij} = s_{ijkl}T_{kl} + d_{kij}E_k, \\ &D_k = d_{kij}T_{ij} + \varepsilon_{kj}E_j, \end{aligned} \tag{16}$$

where e_{ij}, s_{ijkl}, d_{kij}, ε_{ik} are the components of the infinitesimal strain tensor, the compliance tensor, a new piezoelectric tensor and the dielectric permittivity tensor measured at constant stress. The dimensions of these quantities will be discussed in Section 7 in conjunction with some numerical results.

For our purposes it is more convenient to rewrite (16) by choosing stress and electric displacement fields as independent variables. In the case of plane deformations (4) for a piezoelectric crystal 6*mm*, the new equations so obtained can be reduced to the following matrix form (cf., Sosa, 1991):

$$\begin{bmatrix} e_{11} \\ e_{33} \\ 2e_{13} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{33} \\ T_{13} \end{bmatrix} + \begin{bmatrix} 0 & b_{21} \\ 0 & b_{22} \\ b_{13} & 0 \end{bmatrix} \begin{bmatrix} D_1 \\ D_3 \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} E_1 \\ E_3 \end{bmatrix} = -\begin{bmatrix} 0 & 0 & b_{13} \\ b_{21} & b_{22} & 0 \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{33} \\ T_{13} \end{bmatrix} + \begin{bmatrix} \delta_{11} & 0 \\ 0 & \delta_{22} \end{bmatrix} \begin{bmatrix} D_1 \\ D_3 \end{bmatrix},$$

where

$$\begin{aligned}
a_{11} &= \frac{s_{11}^2 - s_{12}^2}{s_{11}} - \frac{d_{31}^2 (s_{11} - s_{12})^2}{s_{11}(s_{11}\varepsilon_{33} - d_{31}^2)}, \\
a_{12} &= \frac{s_{13}(s_{11} - s_{12})}{s_{11}} - \frac{d_{31}(s_{11} - s_{12})(d_{33}s_{11} - d_{31}s_{13})}{s_{11}(\varepsilon_{33}s_{11} - d_{31}^2)}, \\
a_{22} &= \frac{s_{33}s_{11} - s_{13}^2}{s_{11}} - \frac{(d_{33}s_{11} - d_{31}s_{13})^2}{(s_{11}\varepsilon_{33} - d_{31}^2)s_{11}}, \\
a_{33} &= s_{44} - \frac{d_{15}^2}{\varepsilon_{11}}, \quad b_{21} = \frac{d_{31}(s_{11} - s_{12})}{s_{11}\varepsilon_{33} - d_{31}^2}, \quad b_{22} = \frac{d_{33}s_{11} - d_{31}s_{13}}{s_{11}\varepsilon_{33} - d_{31}^2}, \\
b_{13} &= \frac{d_{15}}{\varepsilon_{11}}, \quad \delta_{11} = \frac{1}{\varepsilon_{11}}, \quad \delta_{22} = \frac{s_{11}}{s_{11}\varepsilon_{33} - d_{31}^2},
\end{aligned} \tag{18}$$

which are called the *reduced material constants*. As usual we use the compressed notation for the compliances and the piezoelectric and dielectric permittivity tensors (see, e.g., Eringen and Maugin, 1990; Sosa, 1991). The first six and the last of the nine constants in (18) involve combinations of the mechanical and piezoelectric material properties. Only b_{13} and δ_{11} are purely piezoelectric constants. Thus (17) involves a *full coupling* of mechanical and electrical effects. In writing (17), (18) we took into account that the plane strain conditions imply

$$e_{22} = e_{32} = e_{12} = 0, \quad E_2 = 0, \tag{19}$$

and

$$T_{22} = -\frac{1}{s_{11}}(s_{12}T_{11} + s_{13}T_{33} + d_{31}E_3). \tag{20}$$

Moreover, from the material symmetry properties of the 6*mm* crystals we have

$$D_2 = 0. \tag{21}$$

Finally as a consequence of the positive definiteness of the total energy density

$$2W = 2W(\mathbf{T}, \mathbf{D}) = a_{11}T_{11}^2 + 2a_{12}T_{11}T_{33} + a_{22}T_{33}^2 + a_{33}T_{13}^2 + \delta_{11}D_1^2 + \delta_{22}D_3^2,$$

the reduced material constants satisfy

$$a_{11}, a_{33} > 0, \quad a_{11}a_{22} - a_{12}^2 > 0, \quad \delta_{11}, \delta_{22} > 0. \tag{22}$$

The governing equilibrium equations (1_1), (1_2) become

$$T_{\alpha\beta,\beta} = 0, \quad D_{\alpha,\alpha} = 0. \tag{23}$$

For our purposes (cf., Sosa, 1991), it is convenient to introduce two Airy functions: the *stress function* $U(x_1, x_3)$ and *induction function* $\Psi(x_1, x_3)$ such that

$$T_{11} = U_{,33}, \quad T_{33} = U_{,11}, \quad T_{13} = -U_{,13}, \qquad D_1 = \Psi_{,3}, \quad D_3 = -\Psi_{,1}. \tag{24}$$

Thus the equilibrium equations (23) are automatically satisfied, while the compatibility conditions

$$e_{11,33} + e_{33,11} - 2e_{13,13} = 0, \quad E_{1,3} - E_{3,1} = 0$$

give

$$\begin{aligned} &a_{11}U_{,3333} + (2a_{12} + a_{33})U_{,1133} + a_{22}U_{,1111} - b_{22}\Psi_{,111} - (b_{21} + b_{13})\Psi_{,133} = 0, \\ &b_{22}U_{,111} + (b_{13} + b_{21})U_{,133} + \delta_{22}\Psi_{,11} + \delta_{11}\Psi_{,33} = 0, \end{aligned} \tag{25}$$

provided $U \in C^4(\Sigma)$, $\Psi \in C^3(\Sigma)$. The governing equations (25) are a coupled system of *fourth and third-order* partial differential equations for U and Ψ with coefficients involving the nine reduced constants defined in (18).

4. Problem formulation for a piezoelectric 6*mm* strip

We suppose now that the cross-sectional domain Σ of Section 2 is taken to be the interior of the semi-infinite strip $R = \{(x_1, x_3) : x_1 > 0, -H < x_3 < H\}$. The prescribed tractions t^*_α and surface charge D^* are taken to vanish on the long sides of the strip $x_3 = \pm H$, $x_1 > 0$. Moreover, the prescribed boundary data on the end $x_1 = 0$ are necessarily "self-equilibrated". For this type of loading, it is assumed that the stresses and electric displacement component D_1 decay to zero as $x_1 \to \infty$.

The previous boundary conditions may be written in terms of the Airy functions U, Ψ and then integrated to yield, on using the self equilibration conditions (cf., Horgan and Miller, 1995 for the elastic case):

$$\begin{aligned} &U = 0, \quad U_{,3} = 0, \quad \Psi = 0 \quad \text{at } x_3 = \pm H, \\ &U = f_0(x_3), \quad U_{,1} = f_1(x_3), \quad \Psi = f_2(x_3) \quad \text{at } x_1 = 0, \\ &\int_{-H}^{H} f_0''(x_3)\,dx_3 = 0, \quad \int_{-H}^{H} f_1'(x_3)\,dx_3 = 0, \quad \int_{-H}^{H} f_2'(x_3)\,dx_3 = 0, \\ &U_{,\alpha\beta} \to 0, \Psi_{,\alpha} \to 0 \text{ uniformly in } x_3, \text{ as } x_1 \to \infty, \end{aligned} \tag{26}$$

where f_0, f_1, f_2 are prescribed functions that satisfy suitable smoothness conditions at the corners and the prime denotes differentiation with respect to x_3. Eqs. (25) and (26) constitute the *fundamental boundary-value problem* associated with the semi-infinite 6*mm* piezoelectric strip in a state of plane deformation.

Following the well-known approach extensively used for the purely elastic problem, we seek solutions of equations (25) satisfying (26_1) of the form:

$$U(x_1, x_3) = \exp(-\lambda x_1)F(x_3), \quad \Psi(x_1, x_3) = \exp(-\lambda x_1)G(x_3), \tag{27}$$

where λ is a complex constant and F, G are unknown functions such that

$$F(\pm H) = 0, \quad F'(\pm H) = 0, \quad G(\pm H) = 0. \tag{28}$$

It is important to note that, on using (25), it is easy to prove that in (27) the coefficient of x_1 in the exponential functions must be the *same* for both Airy functions.

On substituting (27) into (25) we obtain

$$\begin{aligned} &a_{11}F^{\mathrm{IV}} + \lambda^2(2a_{12} + a_{33})F'' + \lambda^4 a_{22}F + \lambda^3 b_{22}G + \lambda(b_{21} + b_{13})G'' = 0, \\ &-\lambda^3 b_{22}F - \lambda(b_{13} + b_{21})F'' + \lambda^2\delta_{22}G + \delta_{11}G'' = 0. \end{aligned} \tag{29}$$

Eqs. (29), subject to the boundary conditions (28) constitute an *eigenvalue problem* with eigenparameter λ, for the coupled system of ordinary differential equations for $F(x_3)$, $G(x_3)$. The first of (29) is of *fourth-order* while the second is of *second-order*.

We note that equations (29) can be expressed as

$$\begin{aligned} &L_4F - L_3G = 0, \\ &L_3F + L_2G = 0, \end{aligned} \tag{30}$$

where L_i ($i = 2, 3, 4$) are ordinary differential operators of order 2 and 4 given by

$$\begin{aligned} &L_4 = a_{11}\frac{\mathrm{d}^4}{\mathrm{d}x_3^4} + \lambda^2(2a_{12} + a_{33})\frac{\mathrm{d}^2}{\mathrm{d}x_3^2} + \lambda^4 a_{22}, \\ &L_3 = -\lambda^3 b_{22} - \lambda(b_{21} + b_{13})\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}, \quad L_2 = \lambda^2\delta_{22} + \delta_{11}\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}. \end{aligned} \tag{31}$$

If we eliminate G, Eqs. (30) are reduced to a single *sixth-order* ordinary differential equation for the function F, namely

$$(L_4L_2 + L_3L_3)F = 0, \tag{32}$$

which is

$$\begin{aligned} &a_{11}\delta_{11}F^{\mathrm{VI}} + \lambda^2[a_{11}\delta_{22} + \delta_{11}(2a_{12} + a_{33}) + (b_{21} + b_{13})^2]F^{\mathrm{IV}} + \lambda^4[a_{22}\delta_{11} + \delta_{22}(2a_{12} + a_{33}) \\ &\quad + 2b_{22}(b_{21} + b_{13})]F'' + \lambda^6(a_{22}\delta_{22} + b_{22}^2)F = 0. \end{aligned} \tag{33}$$

We observe that while (33) governs only the mechanical Airy function F, the coefficients involve all nine of the constants defined in (18) and so there is still a coupling of mechanical and electrical effects. Since (33) is a linear homogeneous equation with constant coefficients, its solutions are determined by the roots of the characteristic polynomial equation, i.e.,

$$\begin{aligned} &a_{11}\delta_{11}\omega^6 + \lambda^2[a_{11}\delta_{22} + \delta_{11}(2a_{12} + a_{33}) + (b_{21} + b_{13})^2]\omega^4 + \lambda^4[a_{22}\delta_{11} + \delta_{22}(2a_{12} + a_{33}) \\ &\quad + 2b_{22}(b_{21} + b_{13})]\omega^2 + \lambda^6(a_{22}\delta_{22} + b_{22}^2) = 0. \end{aligned} \tag{34}$$

If we divide both sides of (34) by λ^6 and put $\mu = \omega/\lambda$, then the left hand side of (34) becomes a polynomial $P(\mu)$. This polynomial has no real zeros because we can write

$$P(\mu) = [(b_{21} + b_{13})\mu^2 + b_{22}]^2 + (\delta_{11}\mu^2 + \delta_{22})[a_{11}\mu^4 + (2a_{12} + a_{33})\mu^2 + a_{22}]. \tag{35}$$

Since (22) hold, $P(\mu) > 0, \forall \mu \in \mathbf{R}$ and so the zeros of $P(\mu)$ are complex and conjugate. It is easy to see that the roots of (34) are of the form

$$\begin{aligned} &\omega_1 = \mathrm{i}\lambda\beta_1, \quad \omega_2 = -\mathrm{i}\lambda\beta_1, \quad \omega_3 = \lambda(\alpha_2 + \mathrm{i}\beta_2), \quad \omega_4 = \lambda(\alpha_2 - \mathrm{i}\beta_2), \\ &\omega_5 = \lambda(-\alpha_2 + \mathrm{i}\beta_2), \quad \omega_6 = \lambda(-\alpha_2 - \mathrm{i}\beta_2), \end{aligned} \tag{36}$$

where β_1, α_2, β_2 are real positive constants depending on the properties of the piezoelectric material.

Therefore the general solution of (33) is

$$\begin{aligned} F(x_3) = {} & C_1\cos(\beta_1\lambda x_3) + C_2\sin(\beta_1\lambda x_3) + \mathrm{e}^{\alpha_2\lambda x_3}[C_3\cos(\beta_2\lambda x_3) + C_4\sin(\beta_2\lambda x_3)] \\ & + \mathrm{e}^{-\alpha_2\lambda x_3}[C_5\cos(\beta_2\lambda x_3) + C_6\sin(\beta_2\lambda x_3)], \end{aligned} \tag{37}$$

where C_i, $i = 1, \ldots, 6$ are integration constants. From (29) we can find the expression for G as

$$\begin{aligned} G(x_3) = {} & \lambda^{-3}[\delta_{22}(b_{21} + b_{13}) - b_{22}\delta_{11}]^{-1} \\ & \times \left\{\delta_{11}a_{11}F^{\mathrm{IV}} + \lambda^2[\delta_{11}(2a_{12} + a_{33}) + (b_{21} + b_{13})^2]F'' + \lambda^4[\delta_{11}a_{22} + b_{22}(b_{21} + b_{13})]F\right\}. \end{aligned} \tag{38}$$

where we assume that the coefficient $[\delta_{22}(b_{21}+b_{13})-b_{22}\delta_{11}]$ is non-zero. This certainly holds for the crystals considered in the sequel. For the sake of brevity we omit the explicit expression for G that one obtains on using (37) in (38).

On applying the boundary conditions (28) we can obtain the eigenvalues λ of the problem (28), (29).

5. Eigenconditions for even and odd eigenfunctions

Just as in the purely elastic case for orthotropic or transversely isotropic materials (cf., Choi and Horgan, 1977), it is convenient to study the eigencondition by separating the eigenfunctions into even and odd functions.

Case (A) F an even function.

We note from (38) that G is also an even function. By virtue of $(17)_1$, (24), e_{11} and e_{33} are even functions of x_3, while e_{13} is an odd function so that the deformations are symmetric. Moreover, by virtue of $(17)_2$ and (24), E_1 and E_3 are odd and even functions respectively.

We see that $C_2=0$, $C_5=C_3$, $C_6=-C_4$. On setting $c_1=C_1$, $c_2=2C_3$, $c_3=2C_4$, the function F can be written as

$$F(x_3)=c_1\cos(\beta_1\lambda x_3)+c_2\cos(\beta_2\lambda x_3)\cosh(\alpha_2\lambda x_3)+c_3\sin(\beta_2\lambda x_3)\sinh(\alpha_2\lambda x_3). \tag{39}$$

From (38) and (39) we obtain

$$G(x_3)=\lambda[\delta_{22}(b_{21}+b_{13})-b_{22}\delta_{11}]^{-1}\times\{c_1A_1\cos(\beta_1\lambda x_3)+(c_2A_2+c_3A_3)\cos(\beta_2\lambda x_3)\cosh(\alpha_2\lambda x_3)+(-c_2A_3+c_3A_2)\sin(\beta_2\lambda x_3)\sinh(\alpha_2\lambda x_3)\}, \tag{40}$$

where

$$\begin{aligned}&A_1=\beta_1^4B_1-\beta_1^2B_2+B_3,\quad A_2=B_1(\alpha_2^4-6\alpha_2^2\beta_2^2+\beta_2^4)+B_2(\alpha_2^2-\beta_2^2)+B_3,\\&A_3=2\alpha_2\beta_2[2B_1(\alpha_2^2-\beta_2^2)+B_2],\\&B_1=\delta_{11}a_{11},\quad B_2=\delta_{11}(2a_{12}+a_{33})+(b_{21}+b_{13})^2,\\&B_3=\delta_{11}a_{22}+b_{22}(b_{21}+b_{13}).\end{aligned} \tag{41}$$

On taking into account the boundary conditions (28), after some algebra we arrive at the following eigencondition:

$$\beta_1A_3\sin(\beta_1\lambda H)[\cos(2\beta_2\lambda H)+\cosh(2\alpha_2\lambda H)]+\cos(\beta_1\lambda H)[p_1\sin(2\beta_2\lambda H)+p_2\sinh(2\alpha_2\lambda H)]=0, \tag{42}$$

where

$$p_1=\alpha_2(A_1-A_2)-\beta_2A_3,\quad p_2=\beta_2(A_1-A_2)+\alpha_2A_3. \tag{43}$$

Thus the complex roots of (42) depend on the material constants through β_1, α_2, β_2 defined in (36) and p_1, p_2 defined in (43) and (41).

Case (B) F an odd function.

We note from (38) that G is also an odd function. By virtue of $(17)_1$, (24), e_{11} and e_{33} are odd functions of x_3, while e_{13} is an even function so that the deformations are antisymmetric. Moreover, by virtue of $(17)_2$ and (24), E_1 and E_3 are even and odd functions respectively.

We see that $C_1=0$, $C_5=-C_3$, $C_6=C_4$. On setting $c_1=C_2$, $c_2=2C_3$, $c_3=2C_4$, the function F can be written as

$$F(x_3)=c_1\sin(\beta_1\lambda x_3)+c_2\cos(\beta_2\lambda x_3)\sinh(\alpha_2\lambda x_3)+c_3\sin(\beta_2\lambda x_3)\cosh(\alpha_2\lambda x_3). \tag{44}$$

From (38) and (44) we obtain

$$G(x_3 = \lambda[\delta_{22}(b_{21}+b_{13}) - b_{22}\delta_{11}]^{-1} \\ \{c_1A_1\sin(\beta_1\lambda x_3) + (c_2A_2+c_3A_3)\cos(\beta_2\lambda x_3)\sinh(\alpha_2\lambda x_3) + (-c_2A_3+c_3A_2)\sin(\beta_2\lambda x_3)\cosh(\alpha_2\lambda x_3)\}. \tag{45}$$

On taking into account the boundary conditions (28), after some algebra we arrive at the following eigencondition:

$$\beta_1A_3\cos(\beta_1\lambda H)[\cos(2\beta_2\lambda H) - \cosh(2\alpha_2\lambda H)] + \sin(\beta_1\lambda H)[-p_1\sin(2\beta_2\lambda H) + p_2\sinh(2\alpha_2\lambda H)] = 0. \tag{46}$$

We note that (42) and (46) admit a countable set of complex eigenvalues that appear in symmetric sets of four because if λ is an eigenvalue then $-\lambda, \overline{\lambda}, -\overline{\lambda}$ are also eigenvalues.

Let $L^{\mathrm{e}} = \{\text{solutions } \lambda \text{ to (42)} : \mathrm{Re}\,\lambda > 0\}$, $L^{\mathrm{o}} = \{\text{solutions } \lambda \text{ to (46)} : \mathrm{Re}\,\lambda > 0\}$ and denote by λ_n^{e}, λ_n^{o} respectively those elements of L^{e}, L^{o} lying in the positive quadrant and ordered by increasing real part. Note that

$$L^{\mathrm{e}} = \{\lambda_n^{\mathrm{e}}\}_{n\in N} \cup \{\overline{\lambda}_n^{\mathrm{e}}\}_{n\in N}, \quad L^{\mathrm{o}} = \{\lambda_n^{\mathrm{o}}\}_{n\in N} \cup \{\overline{\lambda}_n^{\mathrm{o}}\}_{n\in N}.$$

At this point, in order to obtain a solution to the problem (25) and (26) we proceed formally. We seek a solution to the problem as two series of eigenfunctions in the form

$$\begin{aligned} U(x_1,x_3) &= \sum_{\lambda^{\mathrm{e}}\in L^{\mathrm{e}}} \exp(-\lambda^{\mathrm{e}}x_1)\xi_{\lambda^{\mathrm{e}}}F_{\lambda^{\mathrm{e}}}(x_3) + \sum_{\lambda^{\mathrm{o}}\in L^{\mathrm{o}}} \exp(-\lambda^{\mathrm{o}}x_1)\eta_{\lambda^{\mathrm{o}}}F_{\lambda^{\mathrm{o}}}(x_3), \\ \Psi(x_1,x_3) &= \sum_{\lambda^{\mathrm{e}}\in L^{\mathrm{e}}} \exp(-\lambda^{\mathrm{e}}x_1)\Xi_{\lambda^{\mathrm{e}}}G_{\lambda^{\mathrm{e}}} + \sum_{\lambda^{\mathrm{o}}\in L^{\mathrm{o}}} \exp(-\lambda^{\mathrm{o}}x_1)\Upsilon_{\lambda^{\mathrm{o}}}G_{\lambda^{\mathrm{o}}}(x_3), \end{aligned} \tag{47}$$

where $F_{\lambda^{\mathrm{e}}}, F_{\lambda^{\mathrm{o}}}, G_{\lambda^{\mathrm{e}}}, G_{\lambda^{\mathrm{o}}}$ are the eigenfunctions corresponding to the eigenvalues $\lambda^{\mathrm{e}}, \lambda^{\mathrm{o}}$ respectively and the coefficients $\xi_{\lambda^{\mathrm{e}}}, \eta_{\lambda^{\mathrm{o}}}, \Xi_{\lambda^{\mathrm{e}}}, \Upsilon_{\lambda^{\mathrm{o}}}$ are to be determined from the data f_0, f_1, f_2.

Here we will not investigate the convergence properties of the formal series (47) (which would require additional suitable restrictions on the data) but focus on our objective of obtaining results on the *rate of exponential decay* of U and Ψ and so of the stresses and D_α. On assuming completeness of the eigenfunctions (see Gregory, 1980a,b for a discussion of this issue for the isotropic elastic case), from (47) and (27) we can conclude that, as $x_1 \to +\infty$, the functions U, Ψ decay as e^{-kx_1} where *k is the real part of the eigenvalue λ with the smallest positive real part*.

6. Asymptotics of the eigenvalues

In this section we find the asymptotic values of the eigenvalues of (42) and (46) with positive real and imaginary parts. We write the eigenvalues as $\lambda_n^{\mathrm{e}} = k_n^{\mathrm{e}} + \mathrm{i}y_n^{\mathrm{e}}$, $\lambda_n^{\mathrm{o}} = k_n^{\mathrm{o}} + \mathrm{i}y_n^{\mathrm{o}}$ respectively. Such results are useful in the numerical solution of (42) and (46) to provide initial estimates for the roots if iterative schemes are used.

First of all we consider the eigenvalue λ_n^{e} and notice that Eq. (42) can be written as

$$\beta_1A_3\sin(\beta_1\lambda_n^{\mathrm{e}}H)[\cos(2\beta_2\lambda_n^{\mathrm{e}}H) + \cos(2\alpha_2\lambda_n^{\mathrm{e}}H\mathrm{i})] + \cos(\beta_1\lambda_n^{\mathrm{e}}H)[p_1\sin(2\beta_2\lambda_n^{\mathrm{e}}H) - \mathrm{i}p_2\sin(2\alpha_2\lambda_n^{\mathrm{e}}H\mathrm{i})] = 0. \tag{48}$$

We use the following asymptotic formulas:

$$\sin(\beta\lambda_n^{\mathrm{e}}H) \sim -\frac{\mathrm{e}^{-\mathrm{i}\beta\lambda_n^{\mathrm{e}}H}}{2\mathrm{i}}, \quad \cos(\beta\lambda_n^{\mathrm{e}}H) \sim \frac{\mathrm{e}^{-\mathrm{i}\beta\lambda_n^{\mathrm{e}}H}}{2}, \tag{49}$$

where β is complex and $f_n \sim g_n$ means that $\lim_{n\to\infty} \frac{f_n}{g_n} = 1$. If we write the asymptotic expressions for the functions in (48) then we obtain

$$e^{-2\lambda_n^e H(\alpha_2+i\beta_2)} \frac{\beta_1 A_3 + p_1}{\beta_1^2 A_3^2 + p_2^2}(\beta_1 A_3 + ip_2) \sim -1. \tag{50}$$

Taking the logarithm we finally obtain the following asymptotic formula for eigenvalues of the symmetric functions:

$$\lambda_n^e \sim \frac{\alpha_2 - i\beta_2}{2H(\alpha_2^2 + \beta_2^2)}[\ln\rho + i\theta + (1+2n)\pi i], \quad n \in Z, \tag{51}$$

where

$$\rho = \frac{|\beta_1 A_3 + p_1|}{\sqrt{\beta_1^2 A_3^2 + p_2^2}} \quad \text{and} \quad \theta = \arctan\frac{p_2}{\beta_1 A_3}$$

and Z denotes the set of all positive or negative integers. From this expression, we find

$$k_n^e \sim \frac{1}{2H(\alpha_2^2 + \beta_2^2)}[\alpha_2 \ln\rho + \beta_2\theta + \beta_2(1+2n)\pi], \quad n \in Z, \tag{52}$$

from which we can calculate the asymptotic decay rate.

Similarly, for antisymmetric deformations, we find that:

$$\lambda_n^o \sim \frac{\alpha_2 - i\beta_2}{2H(\alpha_2^2 + \beta_2^2)}[\ln\rho + i\theta + 2n\pi i], \quad n \in Z, \tag{53}$$

$$k_n^o \sim \frac{1}{2H(\alpha_2^2 + \beta_2^2)}[\alpha_2 \ln\rho + \beta_2\theta + \beta_2 2n\pi], \quad n \in Z, \tag{54}$$

from which we can calculate the asymptotic decay rate.

Since in (52) and (54) the sign of $\alpha_2 \ln\rho + \beta_2\theta$ depends on the particular piezoelectric material being considered, the value of n that minimizes k_n^e, k_n^o cannot be established a priori.

7. Numerical examples

Here we calculate the eigenvalues from the eigenconditions (42) and (46). We consider some piezoelectric ceramics of class 6*mm*: PZT ceramics and Ceramic-B. The material properties of the selected piezoceramics used in the computation are given in Table 1 (see Ruan et al., 2000 and the references cited therein for the values of the material constants).

After calculating the complex roots of (35), we determine the eigenvalues in a very simple manner, i.e., by computing the points of intersection of the curves obtained by equating to zero the real and imaginary parts of (42) and (46). An alternative method would be to use an iterative scheme using the asymptotic results (51) and (53) to provide initial values. The characteristic decay length L is defined as the length over which the Airy functions, and hence the stress and the electric displacement vector, decay to 1% of their values at $x_1 = 0$ and so $L = \frac{\ln 100}{k}$. This provides a measure of the distance from the end beyond which end effects are negligible. In Table 2 the decay rates and decay lengths for the four selected piezoceramics are presented for the case of symmetric and anti-symmetric deformations.

In Table 3, we list the asymptotic values of the decay rates and decay lengths computed from (52) and (54). For the materials under consideration, we have $n = 0$ or $n = -1$. On comparison with the exact values

Table 1
Elastic, piezoelectric and dielectric constants for some piezoceramics

	PZT-5H	PZT-5	PZT-4	Ceramic-B
Elastic compliance (10^{-12} m^2/N)				
s_{11}	16.5	16.4	12.4	8.6
s_{12}	−4.78	−5.74	−3.98	−2.6
s_{13}	−8.45	−7.22	−5.52	−2.7
s_{33}	20.7	18.8	16.1	9.1
s_{44}	43.5	47.5	39.1	22.2
Piezoelectric constant (10^{-12} C/N)				
d_{31}	−274	−172	−135	−58
d_{33}	593	374	300	149
d_{15}	741	584	525	242
Relative permittivity ($\varepsilon_0 = 8.85 \times 10^{-12}$ F/m)				
$\varepsilon_{11}/\varepsilon_0$	1700	1730	1470	1000
$\varepsilon_{33}/\varepsilon_0$	1470	1700	1300	910

Table 2
Decay rate and characteristic decay length for the selected piezoceramics

	Decay rate k (even)	Decay rate k (odd)	Characteristic decay length L (even)	Characteristic decay length L (odd)
PZT-5H	$1.104/H$	$0.436/H$	$2.083 \times 2H$	$5.275 \times 2H$
PZT-5	$1.092/H$	$2.811/H$	$2.106 \times 2H$	$0.818 \times 2H$
PZT-4	$1.337/H$	$2.125/H$	$1.720 \times 2H$	$1.082 \times 2H$
Ceramic-B	$1.016/H$	$2.638/H$	$2.263 \times 2H$	$0.872 \times 2H$

Table 3
Asymptotic decay rates and characteristic decay lengths for the selected piezoceramics

	Decay rate k (even)	Decay rate k (odd)	Characteristic decay length L (even)	Characteristic decay length L (odd)
PZT-5H	$1.864/H$	$0.428/H$	$1.234 \times 2H$	$5.373 \times 2H$
PZT-5	$1.230/H$	$2.610/H$	$1.870 \times 2H$	$0.881 \times 2H$
PZT-4	$1.210/H$	$2.571/H$	$1.900 \times 2H$	$0.895 \times 2H$
Ceramic-B	$1.330/H$	$2.819/H$	$1.730 \times 2H$	$0.816 \times 2H$

given in Table 2, we see that the asymptotic results provide reasonably accurate estimates for their exact counterparts.

8. Conclusions

The decay rates and decay lengths for some engineering piezoceramic materials have been established. The decay lengths for the materials discussed here are *larger* than those of Ruan et al. (2000) where an approximation is introduced in which the authors neglect the last term in Eq. $(25)_1$, the second term in Eq. $(25)_2$ and T_{22} is taken to be equal to zero. The decay lengths are also *larger* than those of *isotropic* elastic materials, for which it is known (see, e.g., Timoshenko and Goodier, 1970, pp. 61–61) that $L(\text{even}) = 1.09$ (2H) while $L(\text{odd}) = 0.61$ (2H). It is worth noting from Table 2 that, *except for* PZT-5H, all the materials considered are such that $L(\text{even}) > L(\text{odd})$, as in the isotropic elastic case. Among the materials considered here, PZT-5H has the largest decay length (for anti-symmetric deformations) and it is over *five* times the strip width. Thus, in general, piezoelectric end effects *penetrate much further into the strip* than their isotropic elastic counterparts, illustrating once again that application of the classical Saint-Venant's principle for isotropic elastic materials must be *significantly modified* when used outside its original context.

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